Growth instabilities in mechanical breakdown under mechanical and thermal stresses

S.-Z. Zhang* and E. Louis
Departamento de Física Aplicada, Universidad de Alicante, Apartado 99, E-03080 Alicante, Spain

O. Plá and F. Guinea
Instituto de Ciencia de Materiales, Consejo Superior de Investigaciones Científicas, Facultad de Ciencias C-III, Universidad Autónoma de Madrid, E-28049 Madrid, Spain

(Received 22 May 1995)

A linear stability analysis is used to investigate crack growth in two dimensional elastic media, and under mechanical or thermal stresses. Although in most cases a circular geometry is considered, the instability of a planar crack is also discussed. Several boundary conditions and size effects are considered. The results indicate that the tendency towards instabilities in mechanical breakdown is stronger than in the case of growth in fields governed by the Laplace equation (diffusion or electrostatic fields), in line with the smaller fractal dimensions obtained in the first case. Instabilities under thermal stresses are shown to depend on the actual thermal gradients. Finally, a model previously investigated numerically is used to show that plasticity decreases the strength of the instability.

PACS number(s): 68.70.+w, 05.40.+j, 61.50.Cj

I. INTRODUCTION

Linear stability analysis has been widely used to illustrate the possibility of instabilities in a large variety of growth problems [1–7]. In particular these instabilities are responsible for the first stages of the formation of the complex structures found in diffusion limited aggregation (DLA) [6] and dielectric breakdown (DB) [8] models. Recently the more complex growth problem of mechanical breakdown [9–12] has also been considered from this point of view in [13–15], and more recently by two of the present authors [16]. In [13] it was proved that a flat crack subjected to tangential forces is unstable to small perturbation, whereas in [16] a circular crack growing in a circular sample was shown to be unstable under the stress fields induced by either constant strain or constant pressure applied at the sample crack surface. In both cases the instability was found to be stronger than in the case of patterns growing in Laplacian fields. The results of Ref. [16] are at variance with those of [14,15] due to omission in the latter works of important terms related to the tensorial nature of the stress field.

In this work we discuss in detail the calculations presented in our previous publication [16], and extend the analysis to several cases of interest, some of which have already been investigated numerically. The paper starts with a brief discussion of the main features of the mechanical breakdown model. Then, the case of a flat crack [13] is analyzed in detail. The instability of a circular crack growing under a variety of boundary conditions is investigated in Sec. IV. Size effects are discussed in two important cases. It is also shown that including a finite threshold stress in the growth law, as done in Refs. [14,15], does not change the response to small perturbations of the circular crack. Instabilities under thermal stresses are discussed in Sec. V. In this case the instability is shown to depend strongly on the local thermal gradients. Finally, the effects of plasticity are discussed in Sec. VI. It is shown that plasticity decreases the strength of the instability, in line with numerical simulations [17].

II. MECHANICAL BREAKDOWN

This growth model, which exhibits a variety of regimes, describes the growth of cracks in an elastic medium [18]. The strain field, $\mathbf{u}(\mathbf{r})$, in an isotropic elastic material of Lamé coefficients λ and μ, satisfies the well-known equilibrium equations [18]:

$$(\lambda+\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u})+\mu\nabla^2\mathbf{u}=0, \tag{1}$$

which is derived from a free-energy density:

$$F=F_0+\lambda\left(\sum_i u_{ii}\right)^2+2\mu\sum_{i,j}u_{ij}^2, \tag{2}$$

where F_0 is the equilibrium free energy and $u_{ij}=\frac{1}{2}(\partial_i u_j+\partial_j u_i)$ is the strain tensor. The stress tensor is defined as, $\sigma_{ij}=\partial F/\partial u_{ij}$.

Within a crack, there are no restoring forces. The force normal to its boundary must vanish, $N_i=\sum_j\sigma_{i,j}n_j=0$, where $\mathbf{n}$ is the vector normal to the boundary.

A simple rule for crack growth, closely related to DLA

*Permanent address: Physics Department, Liaoning University, Shenyang, 110036 People's Republic of China.

and dendritic growth, makes the local velocity of growth at a point on the boundary dependent on the modulus of tansverse stress $\mathbf{T}$ (force) at that point:

$$v_n = f(T). \tag{3}$$

The model can be thought of as a kind of vectorial counterpart of DLA. The role of the diffusive scalar field in DLA is here played by the displacements, $\mathbf{u}$. This model has been studied extensively [9–12], and it resembles DLA in a variety of features. Most notably, when $v_n \propto T$, fractal shapes develop. Setting $v_n \propto T^\eta$, and $\eta \to \infty$, the fractal dimension of the cracks decreases. Dendriticlike patterns can be generated by suppressing the stochastic noise in the growth process [19].

In the following, we will see that, like in other growth models, these structures are associated with linear instabilities in the growth of simple shapes [1]. Cracks grow in the presence of a stress field, induced by external conditions, which can have a different origin. In the following we shall consider some of them.

III. INSTABILITY OF A PLANAR CRACK

This case was discussed in detail in [13]. In that paper the authors obtained the Airy stress function from which they derived the stress tensor [18]. Here we shall follow a completely equivalent analysis and give explicit expressions for the displacement vector and for the stress tensor. Let us consider a rectangular sample of length L (along the y direction) subjected to a tension σ_0 applied at the two boundaries along the x direction ($x = \pm W/2$, where W is the width of the sample). We assume the origin of coordinates to be at the center of the sample. The displacement vector $\mathbf{v}$, which fulfills Lamé's equation and the boundary conditions at the x and y boundaries (note that no stresses should propagate through the boundaries along the y direction) is given by

$$\mathbf{v}(x,y) = [x, -y\nu]\frac{\sigma_0}{E}, \tag{4}$$

where E and ν are the Young's modulus and Poisson's ratio in two dimensions, namely,

$$E = \frac{4\mu(\lambda+\mu)}{(\lambda+2\mu)}, \qquad \nu = \frac{\lambda}{(\lambda+2\mu)}. \tag{5}$$

We now follow the standard linear instability analysis and investigate the stability of the flat crack (for instance, the boundary at $y = y_0 = -L/2$) upon small perturbations of wave number m such as

$$y_p = y_0 + \epsilon e^{imx}, \tag{6}$$

where $\epsilon \ll y_0$. In applying the boundary conditions, we note that the unit vectors normal ($\mathbf{n}$) and tangent ($\mathbf{t}$) to the perturbed surface are given by (after linearizing in ϵ),

$$\mathbf{n} = (-i\epsilon m e^{imx}, 1), \qquad \mathbf{t} = (1, i\epsilon m e^{imx}). \tag{7}$$

As in Ref. [13] we shall assume that L is very large. Then, the displacement vector which fulfills Lamé's equations and the boundary conditions is

$$\mathbf{u}(x,y) = \mathbf{v}(x,y) + \frac{\sigma_0}{2\mu}(im(y-y_0), [1-m(y-y_0)]) \times e^{m[ix-(y-y_0)]} \tag{8}$$

and the components of the stress tensor are

$$\sigma_{xx} = \sigma_0\{1 + m[2 - m(y-y_0)]\epsilon e^{m[ix-(y-y_0)]}\}, \tag{9a}$$

$$\sigma_{yy} = \sigma_0 m^2 (y-y_0)\epsilon e^{m[ix-(y-y_0)]}, \tag{9b}$$

$$\sigma_{xy} = \sigma_{yx} = i\sigma_0 m[1 - m(y-y_0)]\epsilon e^{m[ix-(y-y_0)]}. \tag{9c}$$

As stated above, we now assume that the growth rate is proportional to the modulus of the tangential tension, which in this case is given by (after linearizing in ϵ),

$$\mathbf{T} = \sigma_{xx}(y_p)(1, i\epsilon m e^{imx}). \tag{10}$$

Thus, the instantaneous growth rate can be approximated by

$$\dot{y_0} + \dot{\epsilon} e^{imx} = C\sigma_{xx}(y_p), \tag{11}$$

where C is a constant. Then the ratio between the instantaneous rates of growth of the perturbation ($\dot{\epsilon}$) and that of the flat crack ($\dot{y_0}$) is given by

$$\alpha_m = \frac{\dot{\epsilon}/\epsilon}{\dot{y_0}/y_0} = 2m. \tag{12}$$

This is equivalent to the result reported in Ref. [13] and is twice the one obtained in the case of growth in Laplacian fields [5,6]. This stronger tendency towards instabilities is confirmed in the cases with radial symmetry discussed below.

IV. INSTABILITY OF A CIRCULAR CRACK UNDER CONSTANT STRAIN OR PRESSURE

The free energy in polar coordinates (polar coordinates and a polar reference frame will be hereafter used) is given by

$$F = F_0 + \tfrac{1}{2}\lambda(u_{rr} + u_{\theta\theta})^2 + \mu(u_{rr}^2 + u_{\theta\theta}^2 + 2u_{\theta r}^2) \tag{13}$$

from which the components of the stress tensor can be derived

$$\sigma_{rr} = \frac{E}{1-\nu^2}(u_{rr} + \nu u_{\theta\theta}), \tag{14a}$$

$$\sigma_{\theta\theta} = \frac{E}{1-\nu^2}(\nu u_{rr} + u_{\theta\theta}), \tag{14b}$$

$$\sigma_{r\theta} = \frac{E}{1+\nu}u_{r\theta}, \tag{14c}$$

where E and ν are the Young's modulus and Poisson's ratio in two dimensions given in Eq. (5). It should be noted here that in Ref. [16] the three-dimensional definition of Poisson's ratio was used. The resulting equations and formulas in [16] and those presented here, only differ in some constant factors. The components of the strain tensor are given by

$$u_{rr} = u_r', \;\; u_{\theta\theta} = \frac{\dot{u}_\theta + u_r}{r}, \;\; u_{\theta r} = \frac{1}{2}\left(u_\theta' + \frac{\dot{u}_r - u_\theta}{r}\right), \tag{15}$$

where the partial derivatives with respect to r or θ are denoted by a prime or a dot, respectively. The equilibrium (or Lamé) equations [14] in polar coordinates are given by

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta r}}{\partial\theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \tag{16a}$$

$$\frac{\partial\sigma_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + 2\frac{\sigma_{\theta r}}{r} = 0. \tag{16b}$$

We now consider a circular crack of radius $r = R_1$ growing in an isotropic elastic medium at a rate determined by the boundary conditions to be defined below. What we shall investigate is the stability of this circular crack upon small perturbations of wave number m (m being a positive integer), such as $r_p = R_1 + \epsilon e^{im\theta}$, where $\epsilon \ll R_1$. Once the circular front is perturbed, the most general solution of the Lamé equations can be written as [14,16]

$$u_r(r,\theta) = v_r(r) + \epsilon U_r(r)e^{im\theta}, \tag{17a}$$

$$u_\theta(r,\theta) = i\epsilon U_\theta(r)e^{im\theta}, \tag{17b}$$

where $\mathbf{u}(r,\theta)$ and $v_r(r)$ are the displacement fields in the perturbed and unperturbed cases, respectively. The functions $U_r(r)$ and $U_\theta(r)$ are given by the following expressions:

$$U_r(r) = ar^{1-m} + br^{-1-m} + cr^{m+1} + dr^{m-1}, \tag{18a}$$

$$U_\theta(r) = a\gamma_a r^{1-m} - br^{-1-m} + c\gamma_c r^{m+1} + dr^{m-1}. \tag{18b}$$

The constants γ_a and γ_c in Eqs. (9) are

$$\gamma_a = \frac{4 - m(1+\nu)}{m(1+\nu) + 2(1-\nu)},$$
$$\gamma_c = \frac{4 + m(1+\nu)}{m(1+\nu) - 2(1-\nu)}. \tag{19}$$

In writing Eqs. (18) we have assumed that R_2, the radius of the outer boundary, is not infinitely larger than R_1. In this case the positive powers are not unphysical.

Before imposing the boundary conditions at the crack surface we have to write the unit vectors normal ($\mathbf{n}$) and tangential ($\mathbf{t}$) to the crack surface, which, linearizing in ϵ, take the form

$$\mathbf{n} = \left(1, -\frac{i\epsilon m}{R_1}e^{im\theta}\right), \qquad \mathbf{t} = \left(\frac{i\epsilon m}{R_1}e^{im\theta}, 1\right). \tag{20}$$

The fact that the normal to the surface has a θ component proportional to ϵ was not taken into account in Refs. [14,15]. It should be noted that in the case of Laplacian fields this effect gives a second-order correction that can be neglected. In the present case, and due to the tensorial nature of the field, this correction is of first order and has to be included.

A. Constant strain or pressure at an outer boundary

In both cases, the boundary conditions at the crack surface accounts for the fact that no stresses propagate normal to this surface [18], so that, the two components of the force normal to the surface ($\mathbf{N}$) have to be zero,

$$N_r = \sigma_{rr}n_r + \sigma_{r\theta}n_\theta \approx \sigma_{rr}(r_p) = 0, \tag{21a}$$

$$N_\theta = \sigma_{\theta r}n_r + \sigma_{\theta\theta}n_\theta = \sigma_{\theta r}(R_1) - \frac{i\epsilon m}{R_1}e^{im\theta}\sigma_{\theta\theta}(R_1) = 0. \tag{21b}$$

As regards the outer boundary ($r = R_2$), two boundary conditions are considered, namely, a constant strain (u_0), and a constant pressure (p). In the first case the resulting equations are

$$v_r(R_2) = u_0, \tag{22a}$$

$$U_r(R_2) = U_\theta(R_2) = 0. \tag{22b}$$

Equations (21a) and (22a) give the displacements in the unperturbed case

$$v_r(r) = \beta_{cs}\left[\frac{1-\nu}{1+\nu}r + \frac{R_1^2}{r}\right],$$
$$\beta_{cs} = \frac{u_0(1+\nu)}{(1-\nu)R_2 + \frac{R_1^2}{R_2}(1+\nu)}. \tag{23}$$

On the other hand Eq. (22) combined with Eqs. (21) give the following set of equations for the constants in Eqs. (18),

$$\frac{[m(m+1)-2](1+\nu)}{4\nu - (2+m)(1+\nu)}a - (m+1)R_1^{-2}b + \frac{[m(m-1)-2](1+\nu)}{4\nu - (2-m)(1+\nu)}R_1^{2m}c + (m-1)R_1^{2m-2}d = -\frac{2\beta_{cs}}{R_1^{1-m}}, \tag{24a}$$

$$\frac{m(1-m)(1+\nu)}{4\nu-(2+m)(1+\nu)}a+(m+1)R_1^{-2}b+\frac{m(m+1)(1+\nu)}{4\nu-(2-m)(1+\nu)}R_1^{2m}c+(m-1)R_1^{2m-2}d=\frac{2\beta_{cs}m}{R_1^{1-m}}, \quad (24b)$$

$$a+R_2^{-2}b+R_2^{2m}c+R_2^{2m-2}d=0, \quad (24c)$$

$$\gamma_a a-R_2^{-2}b+\gamma_c R_2^{2m}c+R_2^{2m-2}d=0. \quad (24d)$$

In the case of a constant pressure p_0 at the outer boundary, Eqs. (22) are replaced by

$$\sigma_{rr}(R_2)=p_0, \qquad \sigma_{r\theta}(R_2)=0. \quad (25)$$

As a result the constant β_{cs} in Eq. (23) is replaced by

$$\beta_{cp}=\frac{p_0R_2^2(1+\nu)}{E(R_2^2-R_1^2)}, \quad (26)$$

and the set of equations that gives the four constants in Eqs. (18) has to be modified as follows. In the first two we only need to replace β_{cs} by β_{cp}, whereas the third and the fourth are obtained by replacing in Eqs. (24a) and (24b) R_1 by R_2 and the right-hand side by zero.

As in Refs. [9–11] we assume that the growth rate is proportional to the modulus of the tangential tension ($\mathbf{T}$). This is easily calculated from the stress tensor calculated at the crack surface ($r=r_p$) and the tangential vector given in Eq. (4). The result is

$$\mathbf{T}=\sigma_{\theta\theta}(r_p)\left(\frac{i\epsilon m}{R_1}e^{im\theta},1\right). \quad (27)$$

Thus the instantaneous growth rate is given by

$$\dot{R}_1+\dot{\epsilon}e^{im\theta}=C\sigma_{\theta\theta}(r_p), \quad (28)$$

where C is a constant. Then the result for the ratio between the instantaneous rates of growth of the perturbation ($\dot{\epsilon}$) and that of the circular crack ($\dot{R}_1$) in the case of constant strain is

$$\alpha_m=2(m-1)-\frac{2R_1^{m+1}}{\beta_{cs}}\times\left[\frac{m(m+1)(1+\nu)}{4\nu-(2-m)(1+\nu)}c+(m-1)R_1^{-2}d\right]. \quad (29)$$

In the case of constant pressure β_{cs} should be replaced by β_{cp}, and the constants c and d by those corresponding to this boundary condition. If the outer boundary is placed at the infinite, the result for α_m is

$$\alpha_m=2(m-1) \quad (30)$$

for the two boundary conditions here considered. This result indicates that the circular crack is unstable to perturbations of wave number m greater than 1, as in the case of Laplacian growth [6]. Note, however, that, as in the case of a planar crack, α_m is twice that found in the case of growth in fields governed by the Laplace equation (DLA and DB) [5,6]. This result is in accordance with an analysis of the field singularities along the lines of Ref. [20]. In fact the singularities that appear at wedges in an elastic medium [13,21] are stronger than those found in Laplacian fields [20]. As a consequence the predicted fractal dimensions for the elastic case [10], are smaller than those obtained in Ref. [6] for Laplacian fields, in agreement with numerical results.

We have studied the case of a finite R_2 by numerically solving Eqs. (24) for the constants a–d, and substituting the results for c and d in Eq. (29). The results for α_m are shown in Fig. 1. First some comments on the dependence of the numerical results on the elastic constants are in order. For constant pressure at the outer boundary (first fundamental boundary value problem, see [22]) the numerical results do not depend on the elastic constants. This result is in agreement with a general theorem of plane elasticity theory in an isotropic multiply connected medium [22], which states that for given external stresses and provided that the resultant vectors of the applied forces at the inner boundaries are zero, the stress field does not depend on the elastic constants. In the case of fixed displacements at the outer boundary the results only show a slight dependence on the elastic constants for small R_2/R_1. At large R_2/R_1 this case becomes equivalent to that of constant pressure at the boundary (see above).

The following features of the results of Fig. 1 are worth

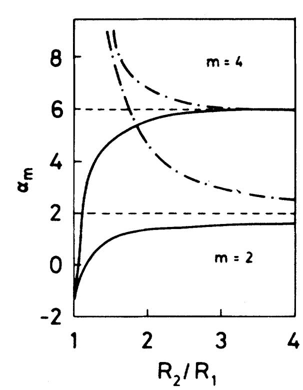

FIG. 1. Results for the ratio between the instantaneous growth rates of the perturbation and that of the circular crack ($\alpha_m, m=2,4$) as a function of R_2/R_1, for $\nu=0.25$ ($\lambda=\mu$) and the two boundary conditions at the outer boundary ($r=R_2$) considered in this work, namely, constant strain (continuous lines) and constant pressure (chain lines). At $R_2=R_1$, the values of α_m for constant strain and pressure are $-4/3$ and ∞, respectively. The horizontal chain lines indicate the values of α_m for the outer boundary at the infinite.

of comment: (i) the results for constant strain are always below the asymptotic value of Eq. (30), whereas the opposite holds for constant pressure, (ii) the asymptotic value is reached faster as m increases, and, (iii) as R_2 tends to R_1, α_m increases up to ∞, in the case of a constant pressure, whereas for constant strain it decreases to $-4/3$, in both cases these values are independent of m. It is interesting to compare these results with those obtained for Laplacian fields. In the latter case α_m is given by [22,23]

$$\alpha_m = m\frac{(R_2/R_1)^{2m} \pm 1}{(R_2/R_1)^{2m} \mp 1} - 1, \tag{31}$$

where $\pm$ signs correspond to fix either the potential or its derivative at the outer boundary (Dirichlet or von Neumann boundary conditions). We note that this equation shows a behavior similar to that found for α_m in the case of elasticity. For instance, for $R_2/R_1 = 1 + \epsilon$, where $\epsilon \ll 1$, α_m tends to either $1/\epsilon - 1$ (for all m) or $m^2\epsilon - 1$, for either constant potential or constant field. We also note that the results for α_m obtained in the case of elasticity (see Fig. 1), are always larger than the values given by Eq. (31) but for R_2 very close to R_1 and fixed strain at the outer boundary. (As an example, we note that for $m = 2$ and $m = 4$ this occurs for $R_2 < 1.05R_1$ and $R_2 < 1.02R_1$, respectively.) Thus, only in a range of R_2 of minor interest, the instabilities in Laplacian fields can be stronger than in the case of elasticity.

B. A pressurized circular crack

We now consider a circular crack in an infinite medium, with an applied pressure inside the crack, p_0. Being the size of the medium infinite, the positive powers of r in the functions $U_r(r)$ and $U_\theta(r)$ are unphysical, and, therefore, the constants c and d have to be zero.

The boundary conditions at the crack surface (r_p) are

$$N_r \approx \sigma_{rr}(r_p) = -p_0, \tag{32a}$$

$$N_\theta = \sigma_{\theta r}(R_1) - \frac{i\epsilon m}{R_1}e^{im\theta}\sigma_{\theta\theta}(R_1) = i\frac{\epsilon m}{R_1}e^{im\theta}p_0, \tag{32b}$$

where we have assumed a hydrostatic pressure inside the hole, and, therefore, perpendicular to the hole surface. These conditions [Eqs. (32)] give the following result for the displacements in the unperturbed case

$$v_r(r) = \frac{p_0(1+\nu)}{E}\frac{R_1^2}{r}. \tag{33}$$

In the perturbed case the following set of equations for the constants a and b in Eqs. (18),

$$\frac{[m(m+1)-2](1+\nu)}{4\nu-(2+m)(1+\nu)}a - (m+1)R_1^{-2}b = -\frac{2p_0(1+\nu)}{ER_1^{1-m}}, \tag{34a}$$

$$\frac{m(1-m)(1+\nu)}{4\nu-(2+m)(1+\nu)}a + (m+1)R_1^{-2}b = \frac{2p_0 m(1+\nu)}{ER_1^{1-m}}. \tag{34b}$$

The result for the ratio between the instantaneous rates of growth of the perturbation and that of the circular crack is

$$\alpha_m = 4(m-1). \tag{35}$$

We note that the tendency towards instability is stronger than in the other two cases discussed above, as could have been easily anticipated.

C. Effects of including a threshold stress in the growth law

If instead of the growth law assumed above, we suppose that there exists a finite material-dependent strength (T_c) that has to be overcome for the crack to propagate, the normal growth velocity will be given by [14,15]

$$v_n = C(T - T_c)^\eta, \tag{36}$$

where, as above, the constant η will be taken equal to 1. As in [14,15], we shall assume that the circular crack is exactly at threshold, this means that the tangential force at the surface of the circular crack is T_c, or equivalently (in the unperturbed case)

$$\sigma_{\theta\theta}(R_1) = T_c. \tag{37}$$

This condition will fix u_0 or p_0 in the cases of constant strain or constant pressure at the outer boundary. As under this requirement the velocity of growth of the circular crack vanishes, α_m cannot be defined. Instead we consider the time evolution of the perturbed surface, namely,

$$r_p = R_1 + \epsilon e^{im\theta+\omega t}. \tag{38}$$

In calculating the growth velocity of the perturbation (ω) we shall assume that the sample is much larger than the crack so that the constants c and d in Eqs. (18) can be neglected. Then, ω is given by

$$B\omega = 2C(m-1)\frac{T_c}{R_1} \tag{39}$$

for both constant strain or pressure at the outer boundary. This result is at variance with the one reported in Refs. [14,15]. The reason is, once again, the components in the tangential and normal unit vectors proportional to ϵ, which were not accounted for in [14,15]. The result is, on the other hand, equivalent to that obtained for the growth law without a threshold, indicating that, from the point of view of growth instabilities, the two models show no differences. A similar conclusion is attained in the case of a pressurized circular crack.

V. INSTABILITY OF A CIRCULAR CRACK UNDER A THERMAL GRADIENT

In this case the free energy density is given by [18]

$$F = F_0(T) - \alpha(\Theta - \Theta_0)(u_{rr} + u_{\theta\theta}) + \frac{1}{2}\lambda(u_{rr} + u_{\theta\theta})^2 + \mu(u_{rr}^2 + u_{\theta\theta}^2 + u_{\theta r}^2), \quad (40)$$

where Θ_0 is a constant temperature, α is a parameter that accounts for the coupling between the elastic and the thermal (Θ) fields. From this free energy the equations of motion can be derived easily.

We shall consider the case of a circular crack of radius R_1 growing in a circular sample, assumed to be much larger than the crack, under the stresses induced by a temperature that varies as a power n of the radial variable r

$$\Theta = \Theta_1 + hr^n \quad \text{for} \quad n < 0, \quad (41)$$

where Θ_1 is a constant temperature. Note that for $n = 0$ no stresses are present in the sample. As in previous cases, the force normal to the crack surface has to be zero. This gives the displacements in the unperturbed case

$$v_r(r) = \frac{\alpha}{2}(\Theta_1 - \Theta_0)r + \frac{\alpha h(1+\nu)}{2(2+n)}\left(r^{1+n} - \frac{R_1^{2+n}}{r}\right) \quad \text{for } n \neq -2, \quad (42a)$$

$$\frac{\alpha}{2}(\Theta_1 - \Theta_0)r + \frac{\alpha h(1+\nu)}{2r}\ln\frac{r}{R_1} \quad \text{for } n = -2. \quad (42b)$$

In considering the perturbed case we note that as we have assumed that the sample diameter is much larger than the crack diameter, the constants c and d in Eqs. (18) can be neglected. The requirement of no forces propagating normal to the crack surface gives the following set of equations for the constants a and b in Eqs. (18),

$$\frac{[m(m+1)-2](1+\nu)}{4\nu - (2+m)(1+\nu)}a - (m+1)R_1^{-2}b = \frac{\alpha h(1+\nu)}{2R_1^{m-n+1}}, \quad (43a)$$

$$\frac{m(1-m)(1+\nu)}{4\nu - (2+m)(1+\nu)}a + (m+1)R_1^{-2}b = -\frac{\alpha h(1+\nu)m}{2R_1^{m-n+1}}. \quad (43b)$$

The result for the ratio between the instantaneous rates of growth of the perturbation and that of the circular crack is

$$\alpha_m = 2(m-1) + n \quad \text{for} \quad n < 0. \quad (44)$$

This result is rather simple and can be easily understood: the instabilities depend on the thermal gradient. On the other hand, the tendency towards instabilities is weaker than for constant strain or constant pressure at the outer boundary (this result may change when a different temperature field is considered).

VI. ELASTIC TO PLASTIC TRANSITION

In this section we discuss the effects of plasticity by means of a model previously used to investigate numerically the effects of plasticity on the fractal dimension of growing cracks [17]. In that model the propagation of a crack in a triangular lattice of classical springs was investigated, assuming that when a bond was broken it became a link between the two lattice nodes which sustains a constant stress (independent of strain) proportional to the stress of the bond at the time of breaking. This behavior is illustrated in Fig. 2, and represents what is known as a perfect elastoplastic medium in the case of a proportionality constant equal to one. It is clear that this model instead of describing crack propagation, concerns a kind of elastic to plastic transition, although it recovers the case of fracture when the proportionality constant vanishes.

Let us consider a disc of radius R_2 subjected to a constant strain at its boundary. Under this condition the radial component of the displacement vector (the only nonzero component) is

$$v_r(r) = u_0\frac{r}{R_2}. \quad (45)$$

According to the model described above, we now assume that the material inside a circle of radius $R_1 < R_2$ undergoes an elastic to plastic transition. The plastic material will sustain a stress, which we will assume to be independent of the strain (see Fig. 2), so that it will no longer propagate stresses. As in [17] we take

$$\sigma_{rr}^{\text{plastic}} = \sigma_{\theta\theta}^{\text{plastic}} = \frac{\delta E}{(1-\nu)}\frac{u_0}{R_2}, \qquad \sigma_{r\theta} = 0 \quad (46)$$

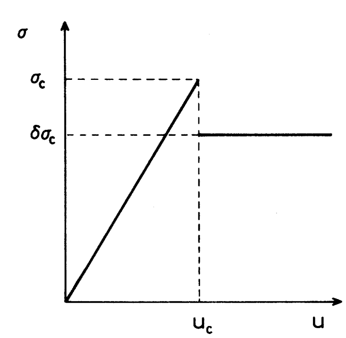

FIG. 2. Stress (σ) vs strain (u) in the model used in this work to investigate the effects of plasticity. When $\delta = 1$ the model is that corresponding to a perfect elastoplastic solid. u_c gives the maximum deformation before plastic behavior occurs.

where δ is a parameter which, as in Ref. [17], is assumed to be less than 1 (Fig. 2).

The displacements in the elastic medium can now be obtained from the two boundary conditions. The first fixes the displacement at the outer boundary

$$v_r(R_2) = u_0, \tag{47}$$

whereas the second accomplishes for the requirement of continuity of the normal forces at the boundary between the two media. These boundary conditions give

$$\sigma_{rr}(R_1) = \frac{\delta E}{(1-\nu)}\frac{u_0}{R_2}, \tag{48}$$

from which the displacements in the elastic medium can be obtained easily:

$$v_r(r) = \beta_{cs}\left[\left(\frac{1-\nu}{1+\nu} + \delta\frac{R_1^2}{R_2^2}\right) r + (1-\delta)\frac{R_1^2}{r}\right], \tag{49}$$

where β_{cs} is the constant given above. This expression reduces to the one found in the case of a circular crack for $\delta = 0$.

Assuming that R_2/R_1 is large enough so that c and d in Eq. (18) can be replaced by zero, an analytic expression for the ratio between the instantaneous rates of growth of the perturbation and that of the circular crack can be obtained. The result is

$$\alpha_m = 2(m-1)\frac{2(1-\delta)(1-\nu)R_2^2}{\delta(1+\nu)R_1^2 + (2-\delta)(1-\nu)R_2^2}. \tag{50}$$

In the case of constant pressure at the outer boundary, the displacements in the presence of a circular crack are given by

$$v_r(r) = \beta_{cp}\left[\frac{1-\nu}{1+\nu}\left(1 - \delta\frac{R_1^2}{R_2^2}\right) r + (1-\delta)\frac{R_1^2}{r}\right], \tag{51}$$

where β_{cp} is given above. This expression reduces to the one found in the case of a circular crack for $\delta = 0$.

The result for the ratio between the instantaneous rates of growth of the perturbation and that of the circular crack is

$$\alpha_m = 2(m-1)\frac{2(1-\delta)R_2^2}{(2-\delta)R_2^2 - \delta R_1^2}. \tag{52}$$

We first note that the explicit dependence of α_m on the elastic constants found for the case of fixed displacements at the outer boundary [Eq. (50)] is a consequence of the boundary condition at the inner boundary [Eq. (48)]. In discussing the behavior of Eqs. (50) and (52) two limits are worthy of comment. If $\delta = 0$ we recover results obtained for standard crack propagation in a circular geometry. Instead if $\delta = 1$, $\alpha = 0$ for all m and the growing pattern should be Eden-like, as expected and found in the numerical simulations of [17]. In the latter case (a perfect elastoplastic medium) our results show that the plastic zone should grow compact, as is likely the case in actual systems, but with a rough surface. For intermediate values of δ, Eq. (52) indicates that a smooth transition from DLA-like to Eden-like is expected to occur, in agreement with [17]. Of course plasticity is a rather complicated phenomenum characterized by the generation of a variety of lattice defects, and is not expected to be described by the simple model discussed here. The present results should be taken as another example of pattern formation.

VII. CONCLUDING REMARKS

In this paper we have presented a study of growth instabilities in mechanical breakdown under mechanical or thermal stresses. For those boundary conditions for which numerical simulations are available we have shown that the instabilities are stronger than for patterns growing in Laplacian fields (DLA or dielectric breakdown), in line with the smaller fractal dimensions obtained in simulations of mechanical breakdown. We have also shown that thermal stresses induce instabilities whose strength strongly depends on the local thermal gradients. We have finally discussed the instabilities in a model previously used to investigate numerically the effects of plasticity. Our results are in complete agreement with the conclusions drawn from the numerical simulations.

ACKNOWLEDGMENTS

We wish to acknowledge financial support from the Spanish CICYT (Grant No. MAT94-0058). S.-Z. Zhang wishes to thank the "Ministerio de Educación y Ciencia" (Spain) for partial support.

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